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On hydromagnetic spin-up

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In this paper we study the generation and propagation of hydromagnetic waves during the spin-up of an electrically conducting, viscous fluid. The time-dependent motion is set up by the impulsive rotation of an infinitely conducting plane which is initially in rigid-body rotation with the fluid, the applied magnetic field being parallel to the axis of rotation. The hydromagnetic flow is permeated by two distinct wave modes and the region between their wave fronts supports harmonic waves produced by the inertial oscillations. The characteristics of these modes are discussed for some limiting cases of interest.

1. Introduction

This paper is concerned with the influence of magnetic forces on the behaviour of a rotating, viscous, incompressible conducting fluid during the spin-up process. It is now well known (Greenspan 1968) that in the absence of external forces the spin-up leads to the formation of an Ekman layer on the parts of the container in contact with the fluid, the final state being achieved through a series of interactions between viscous diffusion and inertial oscillations. In fact (as shown in § 3 of this paper) the spin-up is marked by the propagation of shear-type decaying wave trains generated by the above-mentioned interaction. The triangular interaction between diffusion (viscous and magnetic), inertial oscillations and Alfvén propagation gives rise to a complicated wave system which is the particular concern of the present investigation.

The physical problem described here is similar (but with a change in the boundary conditions on the magnetic field) to the one discussed recently by Benton & Loper (1969). The motivation for reconsidering the problem is based on the fact that their analysis does not reveal adequately the significant role of inertial oscillations in the spin-up process. As a result, the wave character of the flow has been overlooked. Hydromagnetic waves are in fact a dominant feature of the time-dependent interaction for all times and have a crucial bearing on the approach to the ultimate state. In the present investigation, we have derived a relatively simple and complete description of the hydromagnetic flow which is valid for almost all times.

2. Equations of motion and the Laplace transform solution

The fluid is assumed to fill the space between two pole-pieces at $z = 0, h$ of a magnet of strength H_0 which is supposed to be a perfect conductor. The whole system is in rigid-body rotation with angular velocity Ω about an axis normal to the two pole-pieces. At time $t = 0$ the angular velocity of the pole-piece at $z = 0$ is impulsively changed to $\Omega(1+\epsilon)$, where the Rossby number $\epsilon \ll 1$. In this paper we shall be concerned with infinitely separated pole-pieces, that is, $h = \infty$. We take cylindrical polar co-ordinates (r, θ, z) with accompanying fluid velocity $\mathbf{V}$, magnetic field $\mathbf{H}$ and hydromagnetic pressure p. For consistency with the axial symmetry and continuity equations, we define

$$\mathbf{V} = r\Omega\hat{\boldsymbol{\theta}} + \epsilon\Omega[rF_z\hat{\mathbf{r}} + rG\hat{\boldsymbol{\theta}} - 2F\hat{\mathbf{z}}], \tag{2.1}$$

$$\mathbf{H} = H_0\hat{\mathbf{z}} + \epsilon\Omega[-rN_z\hat{\mathbf{r}} - rM\hat{\boldsymbol{\theta}} + N\hat{\mathbf{z}}], \tag{2.2}$$

$$p = \tfrac{1}{2}\rho r^2\Omega^2 + \epsilon\Omega P, \tag{2.3}$$

where F, G, N, M and P are functions of z and t only, ρ is the density and $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{z}}$ are unit vectors in the r, θ and z directions respectively. Substituting (2.1)–(2.3) in the basic hydromagnetic equations and neglecting terms quadratic in ϵ leads to

$$\nu F_{zzz} - F_{zt} + 2\Omega G = (\mu H_0/\rho)\, N_{zz}, \tag{2.4}$$

$$\nu G_{zz} - G_t - 2\Omega F_z = (\mu H_0/\rho)\, M_z, \tag{2.5}$$

$$\eta N_{zzz} - N_{zt} = H_0 F_{zz}, \tag{2.6}$$

$$\eta M_{zz} - M_t = H_0 G_z. \tag{2.7}$$

Here μ, ν and η are the magnetic permeability, kinematic viscosity and magnetic diffusivity of the fluid respectively. Since the surface in contact with the fluid is a perfect conductor, the tangential current must be zero on $z = 0$. Also, all perturbations to the applied magnetic field must vanish at large distances from the pole-piece. Thus the appropriate initial and boundary conditions are

$$F = F_z = G = M = N = N_z = 0 \quad \text{for} \quad z \geqslant 0 \quad (t = 0), \tag{2.8}$$

$$\left.\begin{aligned} F = F_z = 0, \quad G = 1, \quad M_z = N_{zz} = N = 0 \quad \text{on} \quad z = 0 \\ F_z, G, M, N_z \to 0 \quad \text{as} \quad z \to \infty \end{aligned}\right\} \quad \text{for} \quad t \geqslant 0. \tag{2.9}$$

Now we set

$$P = F_z + iG, \quad Q = N_z + iM, \tag{2.10}$$

so that P and Q are given by

$$\nu P_{zz} - P_t - 2i\Omega P = (\mu H_0/\rho)\, Q_z, \quad \eta Q_{zz} - Q_t = H_0 P_z. \tag{2.11}$$

The Laplace transforms of P and Q satisfy the equations

$$\nu P_{zz} - (s + 2i\Omega)\, P = (\mu H_0/\rho)\, Q_z, \quad \eta Q_{zz} - sQ = H_0 P_z, \tag{2.12}$$

with

$$P(0) = i/s, \quad Q_z(0) = 0, \quad P(\infty) = Q(\infty) = 0, \tag{2.13}$$

where s is the transform parameter with respect to t and the same symbols are used for transformed and untransformed quantities. The general solution of the system (2.12) with (2.13) is

$$P = A\,e^{-m_1 z} + B\,e^{-m_2 z}, \quad Q = C\,e^{-m_1 z} + D\,e^{-m_2 z}, \tag{2.14}$$

where

$$m_1, m_2 = \frac{1}{2(\eta\nu)^{\frac{1}{2}}}\left(\{A_0^2 + [(\eta s + 2i\eta\Omega)^{\frac{1}{2}} + (\nu s)^{\frac{1}{2}}]^2\}^{\frac{1}{2}} \pm \{A_0^2 + [(\eta s + 2i\eta\Omega)^{\frac{1}{2}} - (\nu s)^{\frac{1}{2}}]^2\}^{\frac{1}{2}}\right) \tag{2.15}$$

and A_0 is the Alfvén velocity $(\mu H_0^2/\rho)^{\frac{1}{2}}$. The constants of integration A, B, C and D are given by

$$A = \frac{i(s+2i\Omega - \nu m_2^2)}{\nu s(m_1^2 - m_2^2)}, \quad B = \frac{i(s+2i\Omega - \nu m_1^2)}{\nu s(m_2^2 - m_1^2)}, \tag{2.16}$$

$$m_1 C = -m_2 D = -iH_0(s+2i\Omega)/\eta\nu s(m_1^2 - m_2^2). \tag{2.17}$$

3. Spin-up in the non-magnetic case ($\eta = \infty$)

It can be easily shown that the solution in this case is given by (Greenspan 1968, p. 30)

$$P = \tfrac{1}{2}i\{\exp[-z(2i\Omega/\nu)^{\frac{1}{2}}] \times \operatorname{erfc}[z/2(\nu t)^{\frac{1}{2}} - (2i\Omega t)^{\frac{1}{2}}] + \exp[z(2i\Omega/\nu)^{\frac{1}{2}}]\operatorname{erfc}[z/2(\nu t)^{\frac{1}{2}} + (2i\Omega t)^{\frac{1}{2}}]\}. \tag{3.1}$$

This gives a unified representation to the initial Rayleigh flow, the final steady Ekman layers and the decaying inertial oscillations. Expression (3.1) also shows that the spin-up process admits the generation and propagation of diffused waves. To demonstrate this, we make use of a result of Strand (1965) regarding the representation of an error function with complex argument.

Strand (1965) has shown that, for $x > 0$, $y \geqslant 0$,

$$\operatorname{erfc}(x+iy) = e^{-2ixy}\sum_{n=0}^{\infty}(xy)^{2n}[\gamma_n(x) - ixy(n+1)\gamma_{n+1}(x)] \tag{3.2}$$
$$= e^{-2ixy}\phi(x,y) \quad \text{(say)},$$

where

$$\left.\begin{aligned} \gamma_{n+1}(x) &= \frac{2}{(2n+1)\sqrt{\pi}}\left[\frac{e^{-x^2}}{(n+1)!\,x^{2n+1}} - \frac{\sqrt{\pi}}{(n+1)}\gamma_n(x)\right] \quad (n = 0,1,2,3,\ldots), \\ \text{with} \quad \gamma_0(x) &= \operatorname{erfc} x. \end{aligned}\right\} \tag{3.3}$$

Since $\quad \operatorname{erf}[-(x+iy)] = -\operatorname{erf}(x+iy) \quad$ and $\quad \operatorname{erf}(x-iy) = \overline{\operatorname{erf}(x+iy)}$

these cases are also covered by (3.2) and (3.3), but the case $x = 0$ is not. $\phi(x,y)$ is a complex function which tends to zero as $x \to \infty$.

If we now set

$$x = z/2(\nu t)^{\frac{1}{2}} + (\Omega t)^{\frac{1}{2}} \quad \text{and} \quad y = (\Omega t)^{\frac{1}{2}}$$

we can write

$$\operatorname{erfc}\left(\frac{z}{2(\nu t)^{\frac{1}{2}}} + (2i\Omega t)^{\frac{1}{2}}\right) = \exp[-2i(\Omega t + \tfrac{1}{2}z(\Omega/\nu)^{\frac{1}{2}})]\,\phi\left[\frac{z + 2(\nu\Omega t)^{\frac{1}{2}}}{2(\nu t)^{\frac{1}{2}}}, (\Omega t)^{\frac{1}{2}}\right]. \tag{3.4}$$

Thus (3.1) can be expressed as

$$P = \frac{i\exp(-2i\Omega t)}{2}\left\{\exp\left[z\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\right]\phi\left(\frac{z+2(\nu\Omega)^{\frac{1}{2}}t}{2(\nu t)^{\frac{1}{2}}},(\Omega t)^{\frac{1}{2}}\right)-\exp\left[-z\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\right]\right.$$
$$\left.\times\bar{\phi}\left(\frac{|z-2(\nu\Omega)^{\frac{1}{2}}t|}{2(\nu t)^{\frac{1}{2}}},(\Omega t)^{\frac{1}{2}}\right)\right\}+i\exp[-z(2i\Omega/\nu)^{\frac{1}{2}}] \quad \text{for} \quad z < 2(\nu\Omega)^{\frac{1}{2}}t, \tag{3.5a}$$

$$P = \frac{i\exp(-2i\Omega t)}{2}\left\{\exp\left[z\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\right]\phi\left(\frac{z+2(\nu\Omega)^{\frac{1}{2}}t}{2(\nu t)^{\frac{1}{2}}},(\Omega t)^{\frac{1}{2}}\right)+\exp\left[-z\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\right]\right.$$
$$\left.\times\bar{\phi}\left(\frac{|z-2(\nu\Omega)^{\frac{1}{2}}t|}{2(\nu t)^{\frac{1}{2}}},(\Omega t)^{\frac{1}{2}}\right)\right\} \quad \text{for} \quad z > 2(\nu\Omega)^{\frac{1}{2}}t, \tag{3.5b}$$

where $\bar{\phi}(x,y)$ is the complex conjugate of $\phi(x,y)$. Displayed in this form, the function P clearly shows its wave-like behaviour and expresses velocity (and hence vorticity) as diffusion from a source travelling with velocity $2(\nu\Omega)^{\frac{1}{2}}$ away from the rigid container. Equation (3.5a) represents flow behind the wave front $z = 2(\nu\Omega)^{\frac{1}{2}}t$, whereas (3.5$b$) gives the flow ahead of it. However, the wave decays in time of order $\frac{1}{2}\Omega$ through a distance $O[(v/\Omega)^{\frac{1}{2}}]$, the Ekman depth. The first term in (3.5) gives the mirror image of the travelling source. Finally, the last term in (3.5a) yields the ultimate Ekman state.

4. The limit of vanishing (magnetic and viscous) diffusivity

For $\eta \to 0$, $\nu \to 0$ the inverted solution (satisfying the no-slip conditions on $z = 0$) is given by

$$P = i\left[H(t-z/A_0)\,e^{-i\Omega z/A_0}-\Omega z\int_0^t \frac{H(u-z/A_0)\,J_1\{(\Omega/A_0)\,(u^2A_0^2-z^2)^{\frac{1}{2}}\}\,e^{-iu\Omega}}{(u^2A_0^2-z^2)^{\frac{1}{2}}}\,du\right], \tag{4.1}$$

$$Q = (iH_0/A_0)\left[H(t-z/A_0)\,e^{-i\Omega z/A_0}-\Omega\int_0^t H(u-z/A_0)\,e^{-iu\Omega}\right.$$
$$\left.\times\left(\frac{uA_0J_1[(\Omega/A_0)\,(u^2A_0^2-z^2)^{\frac{1}{2}}]}{(u^2A_0^2-z^2)^{\frac{1}{2}}}-iJ_0[(\Omega/A_0)\,(u^2A_0^2-z^2)^{\frac{1}{2}}]\right)du\right], \tag{4.2}$$

$$F = A_0t[\sin(\Omega t)\,J_0(\Omega t)+\cos(\Omega t)\,J_1(\Omega t)]$$
$$-A_0\int_0^t H(u-z/A_0)\sin(u\Omega)\,J_0[(\Omega/A_0)\,(u^2A_0^2-z^2)^{\frac{1}{2}}]\,du, \tag{4.3}$$

where $H(t-z/A_0)$ is the Heaviside unit function and J_0 and J_1 are Bessel functions of the first kind. It can be immediately seen from (4.1) to (4.3) that the perturbed azimuthal and radial fluid motion and field lines are confined to $z \leqslant A_0t$ and that the front $z = A_0t$ of the hydromagnetic wave packets travels with the Alfvén velocity A_0. The region behind the wave front supports a stationary wave, of wavenumber Ω/A_0, which in a weaker sense is circularly polarized; this represents the radial expansion of the Alfvén front. The fluid column ahead of the wave front oscillates purely vertically. In the absence of Coriolis force, $F = 0$ and expressions (4.1) and (4.2) represent Alfvén waves in a non-rotating medium, these being similar to those obtained by Ludford (1959). The splitting of Alfvén waves is a characteristic outcome of the Coriolis–Lorentz force balance.

Now (4.1) can also be expressed as

$$P = i\left[1+\Omega z\int_t^\infty \frac{e^{-iu\Omega}\,J_1[(\Omega/A_0)\,(u^2A_0^2-z^2)^{\frac{1}{2}}]}{(u^2A_0^2-z^2)^{\frac{1}{2}}}\,du\right], \tag{4.4}$$

so that $P\to i$ as $t\to\infty$. Equation (4.3), on the other hand, does not lead to a finite estimate of the steady axial flow. We thus reconsider the Laplace transform of F, which gives the steady-state solution as $F(z,\infty)=\lim_{s\to 0} sF(z,s)$, provided that this limit exists. By taking the limit (with z fixed), we obtain a quasi-steady value of the function F in the form

$$F = \operatorname{Re} iz\operatorname{erfc}\left[\frac{z}{2A_0}\left(\frac{2i\Omega}{t}\right)^{\frac{1}{2}}\right]. \tag{4.5}$$

Expanding this with the help of Strand series (3.2) and (3.3), we get

$$F = z\left\{\sin\frac{\Omega z^2}{2A_0t}\operatorname{erfc}\left[\frac{z}{2A_0}\left(\frac{\Omega}{t}\right)^{\frac{1}{2}}\right]+\frac{z}{A_0}\left(\frac{\Omega}{\pi t}\right)^{\frac{1}{2}}\cos\frac{\Omega z^2}{2tA_0^2}\exp\left(-\frac{\Omega z^2}{2tA_0^2}\right)+o(t^{-1})\right\}. \tag{4.6}$$

It is evident from (4.1)–(4.3) that the initial discontinuity moves away from the pole-piece as an Alfvén wave packet, which is both a current sheet and vortex sheet. At large times, the Alfvén front nears spatial infinity whereas its tail, far behind, is trapped by the gradual inhomogeneity caused by the continual distortion of vortex lines and magnetic field lines; the collapsed wave pattern diffuses with 'induced' diffusivity A_0^2/Ω.

5. The boundary layer and the flow outside

For any fixed $\eta \neq 0$, the boundary-layer solutions are found by taking the limit as ν and z tend to zero while $z/\nu^{\frac{1}{2}}$ remains finite. In the limit $\nu\to 0$

$$\nu^{\frac{1}{2}}m_1 = (A_0^2/\eta+s+2i\Omega)^{\frac{1}{2}},\quad \eta^{\frac{1}{2}}m_2 = \left(\frac{s(s+2i\Omega)}{A_0^2/\eta+s+2i\Omega}\right)^{\frac{1}{2}}, \tag{5.1}$$

such that $(\nu s)^{\frac{1}{2}}$ is not finite. As such, the analysis of this section is not valid for the initial stages of the spin-up process. The Laplace transform of velocity function P within the boundary layer is given by

$$P = \frac{i}{s}\left\{\frac{s+2i\Omega}{A_0^2/\eta+s+2i\Omega}\exp\left[-z\left(\frac{A_0^2+\eta(s+2i\Omega)}{\eta\nu}\right)^{\frac{1}{2}}\right]+\frac{A_0^2}{A_0^2+\eta(s+2i\Omega)}\right\}. \tag{5.2}$$

Inversion gives

$$\begin{aligned}P = {}&\frac{iA_0^2}{A_0^2+2i\eta\Omega}\left\{1-\exp\left[-t\left(\frac{A_0^2}{\eta}+2i\Omega\right)\right]\operatorname{erf}\frac{z}{2(\nu t)^{\frac{1}{2}}}\right\}\\ &-\frac{\eta\Omega}{A_0^2+2i\eta\Omega}\left\{\exp\left[-z\left(\frac{A_0^2+2i\eta\Omega}{\eta\nu}\right)^{\frac{1}{2}}\right]\operatorname{erfc}\left[\frac{z}{2(\nu t)^{\frac{1}{2}}}-\left(\left(\frac{A_0^2}{\eta}+2i\Omega\right)t\right)^{\frac{1}{2}}\right]\right.\\ &\left.+\exp\left[z\left(\frac{A_0^2+2i\eta\Omega}{\eta\nu}\right)^{\frac{1}{2}}\right]\operatorname{erfc}\left[\frac{z}{2(\nu t)^{\frac{1}{2}}}+\left(\left(\frac{A_0^2}{\eta}+2i\Omega\right)t\right)^{\frac{1}{2}}\right]\right\}.\end{aligned} \tag{5.3}$$

Using Strand's results, (3.2) and (3.3), the second term in curly brackets in (5.3) can be written as

$$\{P\} = \exp(-2i\Omega t)\left[\exp[z(R/\nu)^{\frac{1}{2}}\cos\theta]\,\phi\left(\frac{z+2(\nu R)^{\frac{1}{2}}\cos\theta t}{2(\nu t)^{\frac{1}{2}}}, (Rt)^{\frac{1}{2}}\sin\theta\right)\right.$$
$$\left.-\exp[-z(R/\nu)^{\frac{1}{2}}\cos\theta]\,\overline{\phi}\left(\frac{|z-2(\nu R)^{\frac{1}{2}}\cos\theta t|}{2(\nu t)^{\frac{1}{2}}}, (Rt)^{\frac{1}{2}}\sin\theta\right)\right]$$
$$+2[\cos(z(R\nu)^{\frac{1}{2}}\sin\theta)+i\sin(z(R/\nu)^{\frac{1}{2}}\sin\theta)]\exp[-z(R/\nu)^{\frac{1}{2}}\cos\theta]$$
$$\text{for}\quad z<[2(\nu R)^{\frac{1}{2}}\cos\theta]\,t, \tag{5.4a}$$

$$= \exp(-2i\Omega t)\left[\exp[z(R/\nu)^{\frac{1}{2}}\cos\theta]\,\phi\left(\frac{z+2(\nu R)^{\frac{1}{2}}\cos\theta t}{2(\nu t)^{\frac{1}{2}}}, (Rt)^{\frac{1}{2}}\sin\theta\right)\right.$$
$$\left.+\exp[-z(R/\nu)^{\frac{1}{2}}\cos\theta]\,\overline{\phi}\left(\frac{|z-2(\nu R)^{\frac{1}{2}}\cos\theta t|}{2(\nu t)^{\frac{1}{2}}}, (Rt)^{\frac{1}{2}}\sin\theta\right)\right]$$
$$\text{for}\quad z>[2(\nu R)^{\frac{1}{2}}\cos\theta]\,t, \tag{5.4b}$$

where
$$R = (A_0^4/\eta^2+4\Omega^2)^{\frac{1}{2}}, \quad \tan 2\theta = 2\eta\Omega/A_0^2. \tag{5.5}$$

Expression (5.4) corresponds to diffusion from a source of effective diffusivity ν, travelling away from the pole-piece with velocity $2(\nu R)^{\frac{1}{2}}\cos\theta$ (the first term is the mirror image of this source). The effective decay length of the wave is $(\nu/R)^{\frac{1}{2}}\sec\theta$, so that the wave damps out in time of order $(1/2R)\sec^2\theta$. This attenuated hydromagnetic wave system emerges because of the interaction of the diffusing Rayleigh layer both with the inertial oscillations and the applied magnetic field. For $A_0^2/\eta \gg 2\Omega$, the wave front moves with velocity $2A_0(\nu/\eta)^{\frac{1}{2}}$ and the effective damping distance of the diffused waves is the Hartmann depth $(\nu\eta\rho/\mu H_0^2)^{\frac{1}{2}}$. For $A_0^2/\eta \ll 2\Omega$, on the other hand, the velocity of wave propagation is $2(\nu\Omega)^{\frac{1}{2}}$ and the decay length is the Ekman depth $(\nu/\Omega)^{\frac{1}{2}}$. The last term in (5.4*a*) corresponds to the ultimate Hartmann–Ekman state.

For inviscid flow outside the boundary layer, we keep η and z fixed as we let $\nu \to 0$, and obtain

$$P = \frac{iA_0^2}{s(A_0^2+\eta(s+2i\Omega))}\exp\left[-z\left(\frac{s(s+2i\Omega)}{A_0^2+\eta(s+2i\Omega)}\right)^{\frac{1}{2}}\right]. \tag{5.6}$$

An indication of the flow outside the boundary layer is provided by the first term in (5.3), in which the second part sustains up to times of order A_0^2/η. (This term in fact is not negligible outside the inner layer and does affect conditions there.) Expressions (5.6) can thus be simplified further for two cases of interest, for instance, by ignoring A_0^2/η compared with $|s+2i\Omega|$. For $A_0^2/\eta \ll |s+2i\Omega|$, inversion of (5.6) gives

$$P \approx \frac{A_0^2}{2\eta\Omega}\left\{\operatorname{erfc}\frac{z}{2(\eta t)^{\frac{1}{2}}} - \frac{\exp(-2i\Omega t)}{2}\left(\exp\left[z\left(-\frac{2i\Omega}{\eta}\right)^{\frac{1}{2}}\right]\operatorname{erfc}\left[\frac{z}{2(\eta t)^{\frac{1}{2}}}+(-2i\Omega t)^{\frac{1}{2}}\right]\right.\right.$$
$$\left.\left.+\exp\left[-z\left(-\frac{2i\Omega}{\eta}\right)^{\frac{1}{2}}\right]\operatorname{erfc}\left[\frac{z}{2(\eta t)^{\frac{1}{2}}}-(-2i\Omega t)^{\frac{1}{2}}\right]\right)\right\}. \tag{5.7}$$

This can be written as

$$P \approx \frac{A_0^2}{2\eta\Omega}\Bigg\{\operatorname{erfc}\frac{z}{2(\eta t)^{\frac{1}{2}}} - \exp\left[-z\left(\frac{\Omega}{\eta}\right)^{\frac{1}{2}}\right]\left(\cos\left[\left(\frac{\Omega}{\eta}\right)^{\frac{1}{2}} z - 2\Omega t\right] + i\sin\left[\left(\frac{\Omega}{\eta}\right)^{\frac{1}{2}} z - 2\Omega t\right]\right)$$
$$-\frac{1}{2}\left(\exp\left[z\left(\frac{\Omega}{\eta}\right)^{\frac{1}{2}}\right]\bar{\phi}\left(\frac{z+2(\eta\Omega)^{\frac{1}{2}}t}{2(\eta t)^{\frac{1}{2}}}, (\Omega t)^{\frac{1}{2}}\right)\right.$$
$$\left.-\exp\left[-z\left(\frac{\Omega}{\eta}\right)^{\frac{1}{2}}\right]\phi\left(\frac{|z-2(\eta\Omega)^{\frac{1}{2}}t|}{2(\eta t)^{\frac{1}{2}}}, (\Omega t)^{\frac{1}{2}}\right)\right)\Bigg\} \quad \text{for} \quad z < 2(\eta\Omega)^{\frac{1}{2}}t, \tag{5.8a}$$

$$P \approx \frac{A_0^2}{2\eta\Omega}\Bigg[\operatorname{erfc}\frac{z}{2(\eta t)^{\frac{1}{2}}} - \frac{1}{2}\left(\exp\left[z\left(\frac{\Omega}{\eta}\right)^{\frac{1}{2}}\right]\bar{\phi}\left(\frac{z+2(\eta\Omega)^{\frac{1}{2}}t}{2(\eta t)^{\frac{1}{2}}}, (\Omega t)^{\frac{1}{2}}\right)\right.$$
$$\left.+\exp\left[-z\left(\frac{\Omega}{\eta}\right)^{\frac{1}{2}}\right]\phi\left(\frac{|z-2(\eta\Omega)^{\frac{1}{2}}t|}{2(\eta t)^{\frac{1}{2}}}, (\Omega t)^{\frac{1}{2}}\right)\right)\Bigg] \quad \text{for} \quad z > 2(\eta\Omega)^{\frac{1}{2}}t. \tag{5.8b}$$

Interpreted as before, (5.8) expresses P as diffusion from a source (and its mirror image) of diffusity η moving with velocity $2(\eta\Omega)^{\frac{1}{2}}$ through a depth $(\eta/\Omega)^{\frac{1}{2}}$. Moreover, the region between the edge of the viscous layer and the wave front $z = 2(\eta\Omega)^{\frac{1}{2}}t$ supports harmonic electromagnetic waves, of wavenumber $(\Omega/\eta)^{\frac{1}{2}}$, generated by the inertial oscillations of the layer behind.

For $A_0^2/\eta \gg |s+2i\Omega|$, inversion of (5.6) leads to (4.1) and should be interpreted in the same manner.

Now we match the two Laplace solutions (5.2) and (5.6) and integrate to get the axial flow:

$$F = \operatorname{Re}\Bigg\{\frac{i\nu^{\frac{1}{2}}\eta^{\frac{3}{2}}(s+2i\Omega)}{s[A_0^2+\eta(s+2i\Omega)]^{\frac{3}{2}}}\left[1-\exp\left(-z\left[\frac{A_0^2+\eta(s+2i\Omega)}{\eta\nu}\right]^{\frac{1}{2}}\right)\right]$$
$$+\frac{iA_0^2}{s^{\frac{3}{2}}(s+2i\Omega)\,[A_0^2+\eta(s+2i\Omega)]^{\frac{1}{2}}}\left[1-\exp\left(-z\left[\frac{s(s+2i\Omega)}{A_0^2+\eta(s+2i\Omega)}\right]^{\frac{1}{2}}\right)\right]\Bigg\}. \tag{5.9}$$

Evidently the last term becomes indeterminate for $s = 0$ (in the steady state). Thus the quasi-steady solution is given by taking the limit $s \to 0$ (with z fixed). We get (on inversion)

$$F = \operatorname{Re}\Bigg\{\frac{-2\Omega\nu^{\frac{1}{2}}\eta^{\frac{3}{2}}}{(A_0^2+2i\eta\Omega)^{\frac{3}{2}}}\left[1-\exp\left(-z\left(\frac{A_0^2+2i\eta\Omega}{\eta\nu}\right)^{\frac{1}{2}}\right)\right]$$
$$+\frac{izA_0^2}{A_0^2+2i\eta\Omega}\operatorname{erfc}\left[\frac{z}{2}\left(\frac{2i\Omega}{(A_0^2+2i\eta\Omega)\,t}\right)^{\frac{1}{2}}\right]\Bigg\}. \tag{5.10}$$

The first term gives the steady Hartman–Ekman layer whereas the second term shows that the outer region diffuses parabolically with a characteristic diffusion constant $\eta R/\Omega(1+\sin 2\theta)$. This immediately follows if we expand (5.10) by means of Strand series (3.2) and (3.3).

6. Small-time behaviour

For general values of the flow parameters, the small-time behaviour of the flow functions corresponds to large $|s|$. In letting $|s|$ be large, we must keep away from the branch point $s = -2i\Omega$, so that we take $|s+2i\Omega|$ large instead. m_1, m_2,

A and B are then given by

$$\left.\begin{aligned} m_1 &= \left(\frac{s+2i\Omega}{\nu}\right)^{\frac{1}{2}}\left[1+\frac{A_0^2}{2(\eta-\nu)(s+2i\Omega)}+\ldots\right],\\ m_2 &= \left(\frac{s+2i\Omega}{\eta}\right)^{\frac{1}{2}}\left[1-\frac{A_0^2+2i\Omega(\eta-\nu)}{2(\eta-\nu)(s+2i\Omega)}+\ldots\right],\end{aligned}\right\} \tag{6.1}$$

$$\left.\begin{aligned} A &= \frac{i}{s}\left[1-\frac{\eta A_0^2}{(\eta-\nu)^2(s+2i\Omega)}+\ldots\right],\\ B &= \frac{i}{s}\left[\frac{\eta A_0^2}{(\eta-\nu)^2(s+2i\Omega)}+\ldots\right].\end{aligned}\right\} \tag{6.2}$$

Inversion gives

$$\begin{aligned} P = \frac{i}{2}&\left\{\exp\left[z\left(\frac{2i\Omega}{\nu}\right)^{\frac{1}{2}}\right]\operatorname{erfc}\left(\frac{z}{2(\nu t)^{\frac{1}{2}}}+(2i\Omega t)^{\frac{1}{2}}\right)+\exp\left[-z\left(\frac{2i\Omega}{\nu}\right)^{\frac{1}{2}}\right]\right.\\ &\left.\times\operatorname{erfc}\left(\frac{z}{2(\nu t)^{\frac{1}{2}}}-(2i\Omega t)^{\frac{1}{2}}\right)\right\}-\frac{(1-i)zA_0^2}{4(\eta-\nu)(\nu\Omega)^{\frac{1}{2}}}\left\{\exp\left[z\left(\frac{2i\Omega}{\nu}\right)^{\frac{1}{2}}\right]\right.\\ &\left.\times\operatorname{erfc}\left(\frac{z}{2(\nu t)^{\frac{1}{2}}}+(2i\Omega t)^{\frac{1}{2}}\right)-\exp\left[-z\left(\frac{2i\Omega}{\nu}\right)^{\frac{1}{2}}\right]\operatorname{erfc}\left(\frac{z}{2(\nu t)^{\frac{1}{2}}}-(2i\Omega t)^{\frac{1}{2}}\right)\right\}\\ &-\frac{\eta A_0^2}{2\Omega(\eta-\nu)^2}\left\{\exp\left[z\left(\frac{2i\Omega}{\nu}\right)^{\frac{1}{2}}\right]\operatorname{erfc}\left(\frac{z}{2(\nu t)^{\frac{1}{2}}}+(2i\Omega t)^{\frac{1}{2}}\right)+\exp\left[-z\left(\frac{2i\Omega}{\nu}\right)^{\frac{1}{2}}\right]\right.\\ &\times\operatorname{erfc}\left(\frac{z}{2(\nu t)^{\frac{1}{2}}}-(2i\Omega t)^{\frac{1}{2}}\right)-\exp\left[z\left(\frac{2i\Omega}{\eta}\right)^{\frac{1}{2}}\right]\operatorname{erfc}\left(\frac{z}{2(\eta t)^{\frac{1}{2}}}+(2i\Omega t)^{\frac{1}{2}}\right)\\ &-\exp\left[-z\left(\frac{2i\Omega}{\eta}\right)^{\frac{1}{2}}\right]\operatorname{erfc}\left(\frac{z}{2(\eta t)^{\frac{1}{2}}}-(2i\Omega t)^{\frac{1}{2}}\right)\\ &\left.-2\exp(-2i\Omega t)\left(\operatorname{erfc}\frac{z}{2(\nu t)^{\frac{1}{2}}}-\operatorname{erfc}\frac{z}{2(\eta t)^{\frac{1}{2}}}\right)\right\}+\ldots,\end{aligned} \tag{6.3}$$

and a similar expression for Q. The initial impulsive motion immediately causes a Rayleigh layer to develop. This then starts to thicken owing to viscous diffusion. The effect of rotation manifests itself through intertial oscillations and this results in the propagation of the diffusing source with velocity $2(\nu\Omega)^{\frac{1}{2}}$ through the Ekman depth $(\nu/\Omega)^{\frac{1}{2}}$. The distortion of the applied magnetic field by the Rayleigh shear generates electric currents and, owing to the finite electrical conductivity of the fluid, the Rayleigh current layer tends to spread away from the rigid boundary. The interaction of this growing current layer with rotation, again through the inertial oscillations, leads to the propagation of diffused electromagnetic waves with velocity $2(\eta\Omega)^{\frac{1}{2}}$ through the Ekman current layer of thickness $(\eta/\Omega)^{\frac{1}{2}}$. At a later stage, the interplay of these waves and the Alfvén propagation yields a rather complex system of hydromagnetic waves.

7. Asymptotic solutions and approach to the ultimate state

The dominant contributions to P and Q as $t\to\infty$ are associated with the singularities of the field functions at $s=0$ and $-2i\Omega$. Moreover, the final behaviour corresponds to the regions of the complex plane near $s=0$. In order to study the approach to the ultimate state we expand the flow functions in ascending powers of $|s|$, taking $|s|$ and $|s(s+2i\Omega)|^{\frac{1}{2}}$ to be small. The result is

$$m_1 = \left(\frac{A_0^2+2i\eta\Omega}{\nu\eta}\right)^{\frac{1}{2}}(1+\kappa s-\lambda s^2+\ldots), \quad m_2 = \left(\frac{s(s+2i\Omega)}{A_0^2+2i\eta\Omega}\right)^{\frac{1}{2}}(1-\kappa s+\ldots), \tag{7.1}$$

$$\left.\begin{aligned} C &\simeq \frac{2(\eta\nu)^{\frac{1}{2}}\,\Omega H_0}{s(A_0^2+2i\eta\Omega)^{\frac{3}{2}}}+\ldots, \\ D &\simeq \frac{iH_0(s+2i\Omega)^{\frac{1}{2}}}{s^{\frac{3}{2}}(A_0^2+2i\eta\Omega)^{\frac{1}{2}}}\left(1+\frac{2i\eta\nu\Omega s}{(A_0^2+2i\eta\Omega)^2}+\ldots\right), \end{aligned}\right\} \tag{7.2}$$

where
$$\left.\begin{aligned} \kappa &= \frac{(\nu+\eta)\,A_0^2+2i\Omega\eta^2}{2(A_0^2+2i\eta\Omega)^2}, \\ \lambda &= \frac{\eta^2(A_0^2+2i\eta\Omega)^2+6\nu\eta A_0^2(A_0^2+2i\eta\Omega)+\nu^2A_0^2(A_0^2-8i\eta\Omega)}{8(A_0^2+2i\eta\Omega)^4}. \end{aligned}\right\} \tag{7.3}$$

We note that in the above process the flow functions retain the branch points of the original expressions. Inversion of Q and the current function Q_z, after some simplifications, yields

$$\begin{aligned} Q = {} & \frac{2(\eta\nu)^{\frac{1}{2}}\,\Omega H_0}{(A_0^2+2i\eta\Omega)^{\frac{3}{2}}}\left[1-\tfrac{1}{2}\operatorname{erfc}\left(\frac{t-z\kappa[(A_0^2+2i\eta\Omega)/\nu\eta]^{\frac{1}{2}}}{2(\lambda z)^{\frac{1}{2}}}\right)\right]\exp\left[-z\left(\frac{A_0^2+2i\eta\Omega}{\eta\nu}\right)^{\frac{1}{2}}\right] \\ & -\frac{H_0\,e^{-i\Omega t}}{\pi(A_0^2+2i\eta\Omega)^{\frac{1}{2}}}\left(1-\frac{2\eta\nu\Omega^2}{(A_0^2+2i\eta\Omega)^2}\right)K_0\left\{-i\Omega\left(t^2-\frac{z^2}{A_0^2+2i\eta\Omega}\right)^{\frac{1}{2}}\right\} \\ & +\frac{2\eta\nu H_0\,\Omega^2 t\,e^{-i\Omega t}}{\pi(A_0^2+2i\eta\Omega)^2\,(t^2(A_0^2+2i\eta\Omega)-z^2)^{\frac{1}{2}}}K_1\left\{-i\Omega\left(t^2-\frac{z^2}{A_0^2+2i\eta\Omega}\right)^{\frac{1}{2}}\right\} \\ & -\frac{2i\Omega H_0}{\pi(A_0^2+2i\eta\Omega)^{\frac{1}{2}}}\int_0^t e^{-iu\Omega}K_0\left\{-i\Omega\left(u^2-\frac{z^2}{A_0^2+2i\eta\Omega}\right)^{\frac{1}{2}}\right\}du+\ldots, \end{aligned} \tag{7.4}$$

$$\begin{aligned} Q_z = {} & \frac{2\Omega H_0}{A_0^2+2i\eta\Omega}\left\{\left[1-\tfrac{1}{2}\operatorname{erfc}\left(\frac{t-z\kappa[(A_0^2+2i\eta\Omega)/\nu\eta]^{\frac{1}{2}}}{2(\lambda z)^{\frac{1}{2}}}\right)\right]\exp\left[-z\left(\frac{A_0^2+2i\eta\Omega}{\eta\nu}\right)^{\frac{1}{2}}\right]\right. \\ & \left.+\left(1-\frac{i\Omega z}{\pi}\int_t^\infty\frac{e^{-iu\Omega}K_1\{-i\Omega[u^2-(z^2/A_0^2+2i\eta\Omega)]^{\frac{1}{2}}\}}{(u^2(A_0^2+2i\eta\Omega)-z^2)^{\frac{1}{2}}}du\right)\right\}+\ldots, \end{aligned} \tag{7.5}$$

where K_0 and K_1 are modified Bessel functions of the second kind.

Invoking Strands' results, (3.2) and (3.3), we can easily show that the first term in (7.4) represents a diffused hydromagnetic wave propagating through the Hartmann–Ekman layer with velocity V given by

$$V = \frac{2\nu^{\frac{1}{2}}\eta^2R^{\frac{3}{2}}[(\nu+\eta)\,A_0^2\cos 3\theta+2\Omega\eta^2\sin 3\theta]}{(\eta+\nu)^2\,A_0^4+4\Omega^2\eta^4}, \tag{7.6}$$

where R and θ are given by (5.5). For $(\nu/\eta)^{\frac{1}{2}}$ small, the velocity of wave propagation is $2(\nu R)^{\frac{1}{2}}\cos\theta$ (this is the same value as that obtained in § 5), whereas for $A_0^2/\eta \gg 2\Omega$ the wave propagates with velocity $2A_0(\eta\nu)^{\frac{1}{2}}/(\eta+\nu)$. The physical interpretation of the second mode of field functions depends largely upon the behaviour of the modified Bessel functions K_0 and K_1 at large time t. Far away from the pole-piece, we take $(t^2 - z^2A_0^2/\eta^2R^2)$ to be small, but $2\Omega z^2/\eta R^2$ to be large so that we can write

$$e^{-i\Omega t} K_0\left\{-i\Omega\left(t^2 - \frac{z^2}{A_0^2+2i\eta\Omega}\right)^{\frac{1}{2}}\right\} \simeq \left(\frac{\pi i}{2\Omega}\right)^{\frac{1}{2}} \frac{\exp\left[-z(\Omega/\eta)^{\frac{1}{2}}\,\Omega/R\right]}{[t^2-(z^2/A_0^2+2i\eta\Omega)]^{\frac{1}{4}}}$$
$$\times \exp\left[i\Omega\left(\left(\frac{\Omega}{\eta}\right)^{\frac{1}{2}}\frac{z}{R}-t\right)\right]\exp\left[-\frac{R(2i\eta\Omega)^{\frac{1}{2}}}{4z}\left(\frac{z^2A_0^2}{\eta^2R^2}-t^2\right)\right]. \quad (7.7)$$

The field functions thus correspond to a diffused hydromagnetic wave packet consisting of dispersive harmonic waves of wavenumber $R(\eta/\Omega)^{\frac{1}{2}}$ and group velocity $\eta R/A_0$. For $A_0^2/\eta \gg 2\Omega$, the wave velocity is $A_0^2/(\eta\Omega)^{\frac{1}{2}}$ and group velocity is the Alfvén velocity A_0. However, these waves decay through a distance of order $(R/\Omega)\,(\eta/\Omega)^{\frac{1}{2}}$. We find again that the electromagnetic–Coriolis force balance splits the hydromagnetic waves. Rotation is also responsible for the decay of these waves. It is evident from (7.1) that, in the absence of Coriolis force, undamped diffused Alfvén waves propagate in the outer region.

For large t, u in (7.5) is large along the entire path of integration, so we use the asymptotic expansion of K_1 (with z fixed). In the quasi-steady state, (7.5) leads to

$$Q_z = \frac{2\Omega H_0}{A_0^2+2i\eta\Omega}\left\{-\exp\left[-z\left(\frac{A_0^2+2i\eta\Omega}{\eta\nu}\right)^{\frac{1}{2}}\right]+\operatorname{erfc}\left[\frac{z}{2}\left(\frac{2i\Omega}{(A_0^2+2i\eta\Omega)\,t}\right)^{\frac{1}{2}}\right]\right.$$
$$\left.-\frac{z(2i\Omega)^{\frac{1}{2}}}{[\pi t(A_0^2+2i\eta\Omega)]^{\frac{1}{2}}}\exp\left[-\frac{i\Omega z^2}{2t(A_0^2+2i\eta\Omega)}\right]\right\}. \quad (7.8)$$

The first wave mode (of phase velocity V given by (7.6)) decays through the Hartman–Ekman layer and ultimately viscous diffusion and the distortion of vortex lines and magnetic field lines in the inviscid region are balanced. The modified Alfvén wave packet, on the other hand, collapses (owing to the inhomogeneity caused by the above-mentioned distortions) to merge with the magnetic diffusion so that the effective diffusion constant of the growing region now becomes $\eta R/\Omega(1+\sin 2\theta)$ (as shown in § 5).

In the quasi-steady state, expression (5.11) gives the axial velocity at infinity as

$$F(\infty) = -2\nu^{\frac{1}{2}}\Omega\cos 3\theta/R^{\frac{3}{2}}, \quad (7.9)$$

which, for the non-magnetic case ($\eta\to\infty$), yields $F(\infty) = \frac{1}{2}(\nu/\Omega)^{\frac{1}{2}}$. As the strength of the applied magnetic field is increased, the suction velocity decreases and becomes zero when $A_0^2/\eta = 2\Omega/\sqrt{3}$. For $A_0^2/\eta > 2\Omega/\sqrt{3}$, there is an outflow at infinity which reaches a maximum when $A_0^2/\eta = 2\Omega\cot\frac{1}{5}\pi$ and then starts to decrease. For $A_0^2/\eta \gg 2\Omega$, the axial outflow decreases according to

$$F(\infty) \approx -2(\nu\eta)^{\frac{1}{2}}\eta\Omega/A_0^3. \quad (7.10)$$

With the increase in the strength of the applied magnetic field, the inward radial body force, generated by the interaction of the perturbed azimuthal current and the axial field, is very much greater than the perturbed hydrodynamic pressure. The transition from suction to injection at infinity is caused by the 'pinch effect' of this imbalance.

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